

The quark mixing matrix with manifest Cabibbo substructure and an angle of the unitarity triangle as one of its parameters

C. Jarlskog

*Division of Mathematical Physics
LTH, Lund University
Box 118, S-22100 Lund, Sweden*

Abstract

The quark mixing matrix is parameterised such that its "Cabibbo substructure" is emphasised. One can choose one of the parameters to be an arbitrarily chosen angle of the unitarity triangle, for example the angle β (also called Φ_1).

1 Introduction

The question of fermion masses and mixings has been among the most central issues in theoretical particle physics since a long time. Within the three family version of the Standard Model [1] many specific forms for the quark mass matrices have been proposed in the past with the hope that some insight may be gained into the flavour problem. For example, already in 1978 Fritzsch [2] proposed a structure which became quite popular as it could be realised in some Grand Unified Theories (see, for example Ref. [3]). Since then possible zeros in the quark mass matrices (usually called texture zeros) have enjoyed special popularity as these make the computations more transparent and generally lead to specific predictions. Again one has hoped that clues to the solution of the flavour problem may emerge. Another approach has been to "derive" quark mass matrices from experiments, see, for example Ref. [4] where it was found that the two quark mass matrices are highly "aligned".

A troubling factor in all such studies is that the mass matrices are not uniquely defined but are "frame" dependent. In other words, given any set of three-by-three quark mass matrices M_u and M_d , for the up-type and down-type quarks respectively, one can obtain other sets by unitary rotations without affecting the physics. The measurables are, of course, frame-independent and therefore they must be invariant functions under such unitary rotations. These functions were introduced in [5] and studied in detail in [6]. Furthermore, it has been shown recently [7] that this formalism can be extended to the case of neutrino oscillations.

For the quarks what enters, in the standard model, is the pair

$$S_u \equiv M_u M_u^\dagger, \quad S_d \equiv M_d M_d^\dagger \quad (1)$$

The original motivation for the work presented here was to look for "the golden mean" mass matrices, to be defined shortly. First we note that there are two "extreme frames", one in which the up-type quark mass matrix is diagonal, i.e.,

$$S_u = \begin{pmatrix} m_u^2 & 0 & 0 \\ 0 & m_c^2 & 0 \\ 0 & 0 & m_t^2 \end{pmatrix}, \quad S_d = V \begin{pmatrix} m_d^2 & 0 & 0 \\ 0 & m_s^2 & 0 \\ 0 & 0 & m_b^2 \end{pmatrix} V^\dagger \quad (2)$$

where the m 's refer to the quark masses and V is the quark mixing matrix. The other extreme frame is one in which the down-type quark mass matrix is diagonal, i.e.,

$$S_d = \begin{pmatrix} m_d^2 & 0 & 0 \\ 0 & m_s^2 & 0 \\ 0 & 0 & m_b^2 \end{pmatrix}, \quad S_u = V^\dagger \begin{pmatrix} m_u^2 & 0 & 0 \\ 0 & m_c^2 & 0 \\ 0 & 0 & m_t^2 \end{pmatrix} V, \quad (3)$$

One may then wonder how the mass matrices would look like in the "golden mean frame", i.e., the frame right in the middle of the two extremes, where

$$S_u = W^\dagger \begin{pmatrix} m_u^2 & 0 & 0 \\ 0 & m_c^2 & 0 \\ 0 & 0 & m_t^2 \end{pmatrix} W, \quad S_d = W \begin{pmatrix} m_d^2 & 0 & 0 \\ 0 & m_s^2 & 0 \\ 0 & 0 & m_b^2 \end{pmatrix} W^\dagger \quad (4)$$

W is the square root of the quark mixing matrix,

$$V = W^2 \quad (5)$$

In order to go to this frame one needs to compute the square root of the quark mixing matrix. The specific parameterisation of V turns out to be of paramount importance for achieving this goal. In spite of the fact that all valid parameterisations are physically equivalent, most of them are "nasty" and don't allow their roots to be taken so easily. After several attempts and having got stopped by heavy calculations, we have found a particularly convenient parameterisation, presented here below. It turns out that this parameterisation by itself is more interesting than the answer to our original question, which will be dealt with in a future publication.

2 A parameterisation with manifest Cabibbo substructure

The quark mixing matrix is usually parameterised as a function of three rotation angles and one phase, generally denoted by the set $\theta_1, \theta_2, \theta_3$ and δ . However there are many ways in which these parameters can be introduced (for a review see, for example [8]) and the meaning of these quantities depends on how they are introduced. A specific parameterisation may have some beautiful features as well as short-comings. For example, a special feature of the seminal Kobayashi-Maskawa parameterisation [9] is that in the limit $\theta_1 \rightarrow 0$ the first family decouples from the other two. The parameterisation preferred by the Particle Data Group [10] has as its special feature that its phase δ is locked to the smallest angle θ_3 but none of the families decouples if only one of the angles goes to zero. A most important and easy to remember empirical parameterisation has been given by Wolfenstein [13], where the matrix is expanded in powers of a parameter denoted by λ , where $\lambda \simeq 0.22$.

In this article, we introduce an (exact) parameterisation of the quark mixing matrix in terms of four parameters denoted by $\Phi, \theta_3, \delta_\alpha$ and δ_β . The reason for calling one of the angles θ_3 when we have no other θ 's is to stay as close as possible to the usual nomenclature. Our angles δ are often somewhat different from what is commonly used and thus, in order not to confuse the reader, we do not denote them with θ .

We write the quark mixing matrix (exactly) in a form such that its Cabibbo substructure is emphasised from the very beginning,

$$V = V_0 + s_3 V_1 + (1 - c_3) V_2 \quad (6)$$

where $s_3 = \sin\theta_3$, $c_3 = \cos\theta_3$ and the matrices V_j , $j = 0 - 2$, are given by

$$V_0 = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R_2(\Phi) & 0 \\ 0 & 1 \end{pmatrix} \quad (7)$$

$$V_1 = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ b_1^* & b_2^* & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & |A\rangle \\ \langle B| & 0 \end{pmatrix} \quad (8)$$

$$V_2 = \begin{pmatrix} |A\rangle\langle B| & 0 \\ 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (9)$$

Here

$$|A\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad |B\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad (10)$$

and $(|A\rangle\langle B|)_{ij} \equiv a_i b_j^*$. We will impose the following conditions on A and B :

$$\langle A|A\rangle = \langle B|B\rangle = 1 \quad (11)$$

and

$$\begin{aligned} |A\rangle &= -R_2(\Phi)|B\rangle \\ |B\rangle &= -R_2(-\Phi)|A\rangle \end{aligned} \quad (12)$$

By these conditions, the vector A represents two real parameters, for example the magnitude of a_1 and the relative phase of a_1 and a_2 . These will provide the two remaining parameters ($\delta_\alpha, \delta_\beta$) that together with Φ and θ_3 add up to the four parameters needed to get the most general quark mixing matrix. Because of Eq. (12) B introduces no further parameters. Note that

$$V_{13} = a_1 s_3, \quad V_{23} = a_2 s_3, \quad V_{31} = b_1^* s_3, \quad V_{32} = b_2^* s_3 \quad (13)$$

We will also introduce the invariant J defined by

$$\text{Im}(V_{\alpha j} V_{\beta k} V_{\alpha k}^* V_{\beta j}^*) = J \sum_{\gamma, l} \epsilon_{\alpha\beta\gamma} \epsilon_{jkl} \quad (14)$$

In the above parameterisation we find

$$J = s_3^2 c_3 \sin\Phi \cos\Phi \text{Im}(a_1^* a_2) = s_3^2 c_3 \sin\Phi \cos\Phi \text{Im}(b_1^* b_2) \quad (15)$$

where the last equality follows from Eq.(12).

We can check the unitarity of the matrix V without specifying what A (or equivalently B) looks like. We find

$$V_0V_1^\dagger + V_1V_0^\dagger = V_1V_2^\dagger + V_2V_1^\dagger = 0$$

$$V_2V_2^\dagger = V_1V_1^\dagger = -\frac{1}{2}(V_0V_2^\dagger + V_2V_0^\dagger) = \begin{pmatrix} |A\rangle\langle A| & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These identities are derived trivially by using the relation between A and B , Eq.(12).

Given any A or B we have the freedom to rephase it, for example

$$|A\rangle \rightarrow e^{i\eta}|A\rangle \quad (16)$$

whereby the vector B is also rephased by the same amount (see Eq.(12)). From the form of the matrix V we see immediately that the elements $V_{11}, V_{12}, V_{21}, V_{22}$ and V_{33} remain invariant under this rephasing.

In this parameterisation, the usual unitarity triangle, obtained from Eq.(12), is a consequence of

$$a_1\cos\Phi - a_2\sin\Phi + b_1 = 0 \quad (17)$$

Thus the three angles of the triangle are given by the phases of $b_1a_2^*$, $a_2a_1^*$ and $a_1b_1^*$. We can choose our A or B such that one of these angles enters directly as a parameter in the matrix V . The simplest one to incorporate is the angle usually denoted by γ i.e., the phase of $a_2a_1^*$. We could choose

$$|A\rangle = \begin{pmatrix} \sin\delta_\beta e^{-i\delta_\alpha} \\ \cos\delta_\beta \end{pmatrix} \quad (18)$$

whereby

$$\sin\delta_\alpha = \sin\gamma, \quad J = s_3^2 c_3 \sin\Phi \cos\Phi \sin\delta_\beta \cos\delta_\beta \sin\gamma$$

We would then compute B using Eq.(12).

To incorporate the angle β (also denoted by ϕ_1) of the unitarity triangle we could take a_2 to be real and b_1 to have the phase $\delta_\beta = \beta$. From Eq.(12), the reality condition on a_2 implies that $-\sin\Phi b_1 + \cos\Phi b_2$ be real. This fixes the vector B and thereby also the vector A . We find

$$|B\rangle = \frac{1}{\sigma} \begin{pmatrix} \cos\Phi \sin\delta_\alpha e^{i\delta_\beta} \\ -\sin\Phi \sin\delta_\beta e^{-i\delta_\alpha} \end{pmatrix} \quad (19)$$

where

$$\sigma^2 = \cos^2\Phi \sin^2\delta_\alpha + \sin^2\Phi \sin^2\delta_\beta \quad (20)$$

The vector A thus obtained is given by

$$|A\rangle = \frac{1}{2\sigma} \begin{pmatrix} -[\cos 2\Phi \sin(\delta_\alpha + \delta_\beta) + \sin\delta_\alpha e^{i\delta_\beta} - \sin\delta_\beta e^{-i\delta_\alpha}] \\ \sin 2\Phi \sin(\delta_\alpha + \delta_\beta) \end{pmatrix} \quad (21)$$

Here

$$\sin\delta_\beta = \sin\beta(BABAR) = \sin\Phi_1(BELLE) \quad (22)$$

where BABAR [11] and BELLE [12] Collaborations have determined this angle in their study of the $B - \bar{B}$ system but use different notations for it.

With this choice, J is given by

$$J = s_3^2 c_3 \frac{\sin^2(2\Phi) \sin\delta_\alpha \sin\delta_\beta \sin(\delta_\alpha + \delta_\beta)}{4\sigma^2} \quad (23)$$

Finally in order to utilise the third angle, α also known as ϕ_2 , as a parameter we may take it to be the phase of b_1 and require that a_1 be real. The procedure to be followed to achieve this goal is exactly as depicted above.

The above expressions may look somewhat complicated but they are generally quite easy to work with as we often only need their closed forms and not their details.

3 Special features and an estimation of the parameters

The above parameterisation, Eq.(6), is an *exact* form and not a perturbative expansion. It has several special features as follows:

1. In the limit $\theta_3 \rightarrow 0$ the third family decouples from the first two and the exact Cabibbo substructure, with the mixing angle Φ between the first two families, emerges
2. Since the matrices $V_j, j = 0 - 2$, *do not* depend on θ_3 , this parameterisation provides a convenient framework for perturbative expansion in powers of θ_3 which is indeed small, of order λ^2 .
3. We have seen that we can incorporate any one of the angles of the unitarity triangle as one of the four parameters of the mixing matrix.

We now estimate the value of our parameters $\Phi, \theta_3, \delta_\alpha, \delta_\beta$ for the choice Eq.(19) by comparing them with Wolfenstein's parameters [13]. Comparing the matrix elements V_{12} and V_{33} yields that the angles Φ and θ_3 are of order λ and λ^2 respectively

$$\Phi \simeq \lambda, \quad \theta_3 \simeq A\lambda^2 \quad (24)$$

Next, from the moduli of the matrix elements V_{13} , V_{23} , V_{31} , and V_{32} we find that the angle δ_α is much smaller than the angle δ_β ,

$$\sin\delta_\beta \simeq \frac{\eta}{\sqrt{(1-\rho)^2 + \eta^2}} \quad (25)$$

$$\cos\delta_\beta \simeq \frac{1-\rho}{\sqrt{(1-\rho)^2 + \eta^2}} \quad (26)$$

$$(27)$$

$$\sin\delta_\alpha \simeq \eta\lambda^2 \quad (28)$$

Finally, the invariant J is given by

$$J \simeq \theta_3^2 \sin\delta_\alpha = A^2 \lambda^4 \sin\delta_\alpha \quad (29)$$

There is a somewhat subtle issue about this parameterisation that merits to be discussed even though it is hypothetical. It concerns the case with CP conservation while we know that CP is violated and therefore the parameters δ_α and δ_β are both nonvanishing. Nonetheless, we are used to parameterisations with three rotation angles and a phase such that when the phase approaches zero one immediately obtains a mixing matrix with three rotation angles. The converse is not necessarily true that when one of the angles vanishes so does the phase. To remove the phase one often needs to expend some effort. The parameterisation here is more like having two rotation angles and two phases; both of the latter vanish when there is no CP violation. It would seem that we would end up with only two angles, Φ and θ_3 . How do we then recover the third angle, which should be there?

The answer is that even though in the CP conserving limit δ_α and δ_β both approach zero their ratio needs to be defined. We may introduce two angles, θ_1 and θ_2 , by putting

$$\Phi = \theta_1 + \theta_2 \quad (30)$$

$$\frac{\sin\delta_\alpha}{\sin\delta_\beta} = \tan\theta_1 \tan(\theta_1 + \theta_2) \quad (31)$$

Taking the limits carefully as the two δ 's approach zero, we find

$$|B\rangle = \begin{pmatrix} \sin\theta_1 \\ -\cos\theta_1 \end{pmatrix}, \quad |A\rangle = \begin{pmatrix} \sin\theta_2 \\ \cos\theta_2 \end{pmatrix} \quad (32)$$

and thus we end up with a mixing matrix with just three rotation angles. Furthermore, in this limit the invariant J contains three powers of $\sin\delta$ (δ being δ_α or δ_β) in its numerator but only two in its denominator and thus vanishes as it should.

References

- [1] S. L. Glashow, Nucl. Phys. 22 (1961) 579;
S. Weinberg, Phys. Rev. Lett. 19 (1967) 1264;
A. Salam, in Elementary Particle Theory, Ed. N. Svartholm (Almqvist and Wiksell, 1968)
- [2] H. Fritzsch, Phys. Lett. 73B (1978) 317
- [3] H. Georgi and D. V. Nanopoulos, Nucl. Phys. B155 (1979) 52
- [4] P. H. Frampton and C. Jarlskog, Phys. Lett. 154B (1985) 421
- [5] C. Jarlskog, Zeit. f. Phys. C29 (1985) 491; Phys. Rev. Lett. 55 (1985) 1039
- [6] C. Jarlskog, Phys. Rev. D35 (1987) 1685; *ibid* D36 2128
- [7] C. Jarlskog, Phys. Lett. 609B (2005) 323
- [8] C. Jarlskog *in* CP Violation, Ed. C. Jarlskog (World Scientific, 1989) p. 3

- [9] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49 (1973) 652
- [10] Review of Particle Physics, S. Eidelman et al., Phys. Lett B592 (2004) 1
- [11] BABAR Collaboration, B. Aubert et al., Phys. Rev, Lett. 89 201802 (2002)
- [12] BELLE Collaboration, K. Abe et al., Phys. Rev. D66 071102(R) (2002)
- [13] L. Wolfenstein, Phys. Rev. Lett. 51 (1983) 1945